DISPLACEMENT OF POLYDISKS AND LAGRANGIAN FLOER THEORY

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Abstract. There are two purposes of the present article. One is to correct an error in the proof of Theorem 6.1.25 in [FOOO1], from which Theorem J [FOOO1] follows. In the course of doing so, we also obtain a new lower bound of the displacement energy of polydisks in general dimension. The results of the present article are motivated by the recent preprint of Hind [H] where the 4 dimensional case is studied. Our proof is different from Hind's even in the 4 dimensional case and provides stronger result, and relies on the study of torsion thresholds of Floer cohomology of Lagrangian torus fiber in simple toric manifolds associated to the polydisks.

Contents

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1. Introduction

In [FOOO1], we stated a *lower bound* of the displacement energy of relatively spin Lagrangian submanifold L with a bounding cochain b in terms of torsion exponents of Floer cohomology $HF((L,b),(L,b);\Lambda_{0,\mathrm{nov}})$. This is Theorem J in [FOOO1], which is a consequence of Theorem 6.1.25 in the book. See Theorem 6.1 and Theorem 6.2 of this paper for the precise statement in a general setting. However, the proof of Theorem 6.1.25 contains an error. One of the purposes of the present paper is to correct this error. There are two key ingredients in this correction of the error, which are interrelated to each other: one is the usage of an optimal Floer chain map introduced in the present paper, and an energy estimate of the type which was first introduced by Chekanov [Che] and clarified by the second named author in [Oh3].

Another purpose is to employ Theorem J [FOOO1] and obtain an estimate of the lower bound of the displacement energy of polydisks in cylinder which provides a higher dimensional generalization of a recent result of Hind [H] as well as an improvement of Hind's result.

Before we achieve the two purposes above, we prove the following non-displacement theorem to illustrate a geometric consequence of Theorem J [FOOO1] in a simple example. We denote by S the *Gromov width* of any domain of our interest. For example, the ball $B^{2k} \subset \mathbb{C}^k$ of radius $r > 0$ has Gromov width

$$S = \pi r^2.$$

Let $S^1(S) \subset \mathbb{C}$ be a circle of radius $r = \sqrt{\pi^{-1}S}$, i.e., $S = \pi r^2$. We also consider $S^1_{eq} \subset S^2(1)$ the equator in the sphere of area 1.

Theorem 1.1. *We put $X = \mathbb{C} \times S^2(1)$. Suppose $S > 1/2$. Consider any time-dependent Hamiltonian $H : [0,1] \times X \to \mathbb{R}$ with its Hofer norm $\|H\| < S$. Then we have*

$$\psi_H(S^1(S) \times S^1_{eq}) \cap (S^1(S) \times S^1_{eq}) \neq \emptyset$$

for its time-one map $\psi_H := \phi^1_H$.

The proof of this theorem is 'elementary' in that it uses only the Lagrangian Floer theory for monotone Lagrangian submanifolds [Oh1] and by now standard computations for the energy estimates used as in [Che], [Oh3], but does *not* use any techniques of virtual fundamental chains, Bott-Morse theory or any higher homological algebra.

In fact, this theorem is a corollary of Theorem J [FOOO1] whose precise statement we refer to Section 6. We provide this elementary proof in this particular case partly because it nicely illustrates Theorem J [FOOO1] when a deformation of Floer cohomology by a bounding cochain is not needed. (On the other hand, the proof of Theorem J [FOOO1] is given in a very general context in terms of the deformed Floer cohomology of *weakly unobstructed* Lagrangian submanifolds *after bulk deformations*.)

Remark 1.2. Another motivation for us to prove this particular theorem is related to the *upper bound* in the following inequality stated in [H]:

$$\left(\frac{1}{2} - \varepsilon\right) \lfloor S \rfloor + \varepsilon \leq e^{Z_{1,1}}(D(1,S)) \leq \frac{S}{2} + 3 \tag{1.1}$$

where $e^{Z_{1,1}}(D(1,S))$ is the displacement energy (Definition 2.1 (2)) of the bidisk $D(1,S)$ in the cylinder $Z_{1,1} = Z_{1,1}(1+\varepsilon)$. Here, following the notation of [H], we denote the bidisks in $\mathbb{C}^2$ by

$$D(a,b) = \{(z_1,z_2) \mid \pi|z_1|^2 < a,\, \pi|z_2|^2 < b\} = D^2(a) \times D^2(b) \subset \mathbb{C}^2$$

with $a \le b$, and also denote the cylinder in $\mathbb{C}^2$ by

$$Z_{1,1}(a+\varepsilon) = \{(z_1,z_2) \mid \pi|z_1|^2 < a+\varepsilon\} = D^2(a+\varepsilon) \times \mathbb{C}$$

for $\varepsilon > 0$ small. Hind's proof of the upper bound uses an explicit construction of displacing Hamiltonian isotopy. However, the construction used in his proof seems to directly contradict to the above Theorem 1.1, and also to Theorem J [FOOO1]. In fact, Theorem J [FOOO1] implies the following stronger lower bound

$$S \le e^{Z_{1,1}}(D(1,S)) \tag{1.2}$$

whenever $S > 1$.

Hind [H] obtained his lower bound in (1.1) by using some embedding obstruction arising from an explicit study of moduli space of proper holomorphic curves in $S^2 \times S^2 \setminus E$ where E is the image of certain symplectic embedding of ellipsoid. His study on this lower bound heavily relies on the compactification result in symplectic field theory [Ho2], [BEHWZ]. Partly because such an explicit study of moduli space in high dimension is not available, Hind restricts himself to the 4 dimensional case. Our proof is different from Hind's and relies on the study of *torsion thresholds* of Floer cohomology of Lagrangian torus fiber and Theorem J [FOOO1].

However, although the statement of Theorem J [FOOO1] is correct as it is, its proof contains some incorrect argument at the end of the proof of Theorem 6.1.25 in p.392: *The homomorphisms in line 9 and 11 of page 392 of* [FOOO1] *is not well-defined.* And to give a correct proof of the same statement as stated in Theorem 6.1.25, hence Theorem J, we also need to improve the energy estimate given in Proposition 5.3.45 [FOOO1] and use a different construction of a Floer chain map. It turns out that to obtain the the optimal energy estimate needed to prove Theorem J in the construction of a Floer chain map, we need to use the Hamiltonian perturbed Cauchy-Riemann equation with *fixed* Lagrangian boundary condition which intertwines the so called the geometric version of the Floer cohomology and the dynamical version of the Floer cohomology and then applying the *coordinate change* that relates the two. Similar coordinate change was used by the second named author previously in [Oh2] for a similar purpose. Using this trick and an optimal energy estimate originated by Chekanov [Che], we can prove the statement of Theorem 6.1.25 and hence Theorem J in [FOOO1] as they are currently stated.

Another purpose of the present paper is to apply Theorem J [FOOO1] in the study of displacement energy of polydisks in arbitrary dimension and generalize the above mentioned 4 dimensional result to arbitrary dimension whose description is now in order. It turns out that the Lagrangian Floer theory developed in [FOOO1] and [FOOO2, FOOO3] can be nicely applied to the various symplectic topological questions concerning polydisks $D^2(r_1) \times \cdots \times D^2(r_n)$. This is largely because the polydisks contain Lagrangian tori which can be embedded into the toric manifolds $S^2(a_1) \times \cdots \times S^2(a_n)$ or $S^2(a) \times \mathbb{C}P^{n-1}(\lambda)$ for suitable choices of a_i's or (a,λ). Here $\mathbb{C}P^{n-1}(\lambda)$ is the projective space with the Fubini-Study Kähler form ω with $[\omega](C) = \lambda$ for the homology class C of the complex line. With this notation, we have the symplectic embedding of $B^{2n}(\lambda) \hookrightarrow \mathbb{C}P^{n-1}(\lambda)$ such that $\mathbb{C}P^{n-1}(\lambda) \setminus$

$B^{2n}(\lambda)$ is the hyperplane at infinity. In this way, we obtain various improvements and generalizations of the theorems concerning symplectic topology of polydisks proven in [HK], [H].

We give two high dimensional generalizations of (1.2). Denote by $(z_1, \dots, z_n)$ the complex coordinates of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. We decompose

$$(z_1, \dots, z_n) = (z_1, z')$$

with $z' = (z_2, \dots, z_n)$. We denote

$$D(a_1, a_2, \dots, a_n) = \{(z_1, \dots, z_n) \mid \pi |z_1|^2 < a_1, \dots, \pi |z_n|^2 < a_n\} \subset \mathbb{C}^n$$

where $a_1 \le a_2 \le \cdots \le a_n$. Hind [H] considers only the case when $n = 2$. We also denote the cylinder over the disk $|z_1|^2 \le (a_1 + \varepsilon)/\pi$ by

$$Z_{1,n-1}(a_1 + \varepsilon) = \{(z_1, \dots, z_n) \mid \pi |z_1|^2 < a_1 + \varepsilon\}$$

for $\varepsilon > 0$ small. The following two theorems can be regarded as two different high dimensional generalizations of the lower bound in (1.1).

Theorem 1.3. *Let $S > 1$ and $0 < \varepsilon < 1$. Put $Z_{1,n-1} = Z_{1,n-1}(1 + \varepsilon)$. Then we have*

$$S \le e^{Z_{1,n-1}}(D(1, S, \dots, S)).$$

Theorem 1.4. *Let $S > 1$ and $0 < \varepsilon < 1$. Let k be an integer satisfying $1 \le k < n$. Put $Z_{n-k,k} = Z_{n-k,k}(1 + \varepsilon) = D^2(1 + \varepsilon)^{n-k} \times \mathbb{C}^k$. Then we have*

$$S \le e^{Z_{n-k,k}}(D^2(1)^{n-k} \times B^{2k}(kS)).$$

This paper borrows many notations and definitions from [FOOO1] without delving into detailed explanations thereof. Especially the notion of bulk deformations is used mainly to make the statement of Theorem J from [FOOO1] in this paper as close as possible to that of [FOOO1]. We refer to the relevant sections of [FOOO1] for more explantions thereof. However, for those who are mainly interested in the overall argument how the torsion threshold can be used in the study of displacement energy, we recommend them to directly look at Section 3, Section 7 and Section 8 and refer to other sections as needed.

In March 2010, R. Hind gave a talk at MSRI workshop on Symplectic and Contact Topology and Dynamics: Puzzles and Horizons. Thanks to his talk, we took another look at our proof of Theorem 6.1.25 in [FOOO1] and found out an inaccurate point in the proof, which we rectify in this paper. We thank him for his interesting talk and discussion.

2. Notations

We introduce the following general definitions and notations which we will use in this paper.

Let (X, ω) be a symplectic manifold. We denote by J a time-dependent family of ω-compatible almost complex structures $J = \{J_t\}_{t \in [0,1]}$.

Let $\psi : X \to X$ be a Hamiltonian diffeomorphism and $H \in C^\infty([0,1] \times X)$ a *normalized* Hamiltonian with $\phi_H^1 = \psi$ and $\int_X H_t \, \omega^n = 0$. We denote the Hofer norm (see [Ho1]) of H by

$$\|H\| = \int_0^1 (\max H_t - \min H_t) \, dt. \tag{2.1}$$

We define the Hofer norm of ψ by

$$\|\psi\| = \inf_{H \mapsto \psi} \|H\|, \tag{2.2}$$

where $H \mapsto \psi$ means that $\psi = \phi_H^1$. We also define the length of a Hamiltonian isotopy $\phi_H = \{\phi_H^t\}$ by

$$\mathrm{leng}(\phi_H) = \|H\|. \tag{2.3}$$

Following Weinstein's notation [W], we denote the set of Hamiltonian deformations of L by

$$\mathfrak{Iso}(L) = \{\psi(L) \mid \psi \in Ham(X, \omega)\}$$

for a given Lagrangian submanifold $L \subset (X, \omega)$.

Definition 2.1. (1) Let $L' \in \mathfrak{Iso}(L)$. Then we define the *Hofer distance* between L, L' by

$$\mathrm{dist}(L, L') = \inf_{H \in C^\infty([0,1]\times X)} \{\|H\| \mid \phi_H^1(L) = L'\}.$$

Or equivalently,

$$\mathrm{dist}(L, L') = \inf_{\psi \in Ham(X,\omega)} \{\|\psi\| \mid \psi(L) = L'\}.$$

(2) Let $Y \subset X$. We define the *displacement energy* $e^X(Y) \in [0, \infty]$ by

$$e^X(Y) := \inf_{\psi \in Ham(X,\omega)} \{\|\psi\| \mid \ \psi(Y) \cap \overline{Y} = \emptyset\}.$$

We put $e^X(Y) = \infty$ if there exists no $\psi \in Ham(X, \omega)$ with $\psi(Y) \cap \overline{Y} = \emptyset$. When no confusion can occur, we simply write $e(Y)$ instead of $e^X(Y)$.

Let $\rho(\tau)$ be a smooth function on $\mathbb{R}$ such that $\rho(\tau) = 0$ or 1 when $|\tau|$ is sufficiently large. In this paper we will take $\rho(\tau)$ as one of the following three types, either

$$\begin{aligned} \rho_+(\tau) &= \begin{cases} 0 & \text{for } \ \tau \leq 0 \\ 1 & \text{for } \ \tau \geq 1 \end{cases} \\ \rho_+'(\tau) &\geq 0 \end{aligned} \tag{2.4}$$

or $\rho_- = 1 - \rho_+$ or ρ_K satisfying

$$\begin{aligned} \rho_K(\tau) &= \begin{cases} 0 & \text{for } \ |\tau| \geq K \\ 1 & \text{for } \ |\tau| \leq K-1 \end{cases} \\ \rho_K' &\geq 0 \quad \text{on } \ [-K, -K+1], \quad \rho_K' \leq 0 \quad \text{on } \ [K-1, K] \end{aligned} \tag{2.5}$$

for $K \geq 1$ and ρ_K goes down to $\rho_{K=0} = 0$, e.g., $\rho_K = K\rho_1$. (See (5.7).)

Ordering of the arguments in the pair $(L^{(0)}, L^{(1)})$ varies in the notations from [FOOO1] for the various objects associated to the pair of Lagrangian submanifold $(L^{(0)}, L^{(1)})$. We mostly follow them in the present paper. Specifically, we would like to mention the following conventions:

1. (Path spaces) $\Omega(L^{(0)}, L^{(1)}; \ell_a)$.
2. (Floer moduli spaces) $\mathcal{M}(L^{(1)}, L^{(0)})$
3. (Floer complex and homology) $CF(L^{(1)}, L^{(0)})$, $HF(L^{(1)}, L^{(0)})$.

3. A non-displacement theorem

In this section, we prove Theorem 1.1. Recall that $S^1(S) \subset \mathbb{C}$ is a circle of radius r with its area $\pi r^2 = S$ and $S^1_{eq} \subset S^2(1)$ is the equator in the sphere of area 1. We put $X = \mathbb{C} \times S^2(1)$.

Proof of Theorem 1.1. Suppose to the contrary that

$$\psi_H(S^1(S) \times S^1_{eq}) \cap (S^1(S) \times S^1_{eq}) = \emptyset \tag{3.1}$$

for a Hamiltonian H with $\|H\| < S$. We denote

$$c := S - \|H\| > 0.$$

Let $L^{(0)} = S^1(S) \times S^1_{eq}$. We choose $L^{(1)}$ as a small Hamiltonian perturbation of $S^1(S) \times S^1_{eq}$ defined as follows: We move $S^1(S)$ and S^1_{eq} by small isometries on $\mathbb{C}$ and $S^2(1)$ respectively and obtain $S^1_2(S)$, $S^1_{2,eq}$ so that

$$\psi_H(L^{(0)}) \cap (L^{(1)}) = \emptyset$$

where we denote $L^{(1)} := S^1_2(S) \times S^1_{2,eq}$.

Since the condition (3.1) is an open condition, such an isometry obviously exists. We choose this perturbation so that

$$\mathrm{dist}(L^{(0)}, L^{(1)}) = \mathrm{dist}(S^1(S) \times S^1_{eq}, S^1_2(S) \times S^1_{2,eq}) \leq \frac{c}{4}. \tag{3.2}$$

We then will deduce contradiction. We remark that $L^{(0)} \cap L^{(1)}$ consists of four transversal intersection points. We take a one-parameter family of smooth functions ρ_K on $\mathbb{R}$ satisfying (2.5). We denote by X_H the Hamiltonian vector filed of H defined by $dH = \omega(X_H, \cdot)$.

Consider any pair (p_-, p_+) of intersection points in $L^{(0)} \cap L^{(1)}$ and the equation

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - \rho_K(\tau) X_H(u)\right) = 0 \tag{3.3}$$

of $u = u(\tau, t) : \mathbb{R} \times [0,1] \to \mathbb{C} \times S^2$ satisfying the boundary condition

$$u(\tau, 0) \in L^{(0)},\ u(\tau, 1) \in L^{(1)},\ u(\pm\infty, t) \equiv p_\pm \in L^{(0)} \cap L^{(1)} \tag{3.4}$$

and the finite energy condition

$$E_{(J,H,\rho_K)}(u) := \frac{1}{2} \int_{\mathbb{R} \times [0,1]} \left|\frac{\partial u}{\partial \tau}\right|^2 + \left|\frac{\partial u}{\partial t} - \rho_K(\tau) X_H(u)\right|^2 d\tau dt < \infty. \tag{3.5}$$

Here we use the canonical complex structure J on $\mathbb{C} \times S^2(1)$, which we do not perturb. Note that any solution u of (3.3), (3.4) carries a natural homotopy class. We denote this homotopy class by B. As standard in Floer theory we define the equivalence relation $B \sim B'$ if and only if

$$\omega(B_1) = \omega(B_2), \quad \mu(B_1) = \mu(B_2) \tag{3.6}$$

and denote by $\Pi(p_-, p_+)$ the set of equivalence classes. Here μ denotes the Maslov index of the map u associated to the pairs $(L^{(0)}, L^{(1)})$ of Lagrangian submanifolds and the asymptotic condition $p_\pm$. (We refer to Section 2.2 [FOOO1] for a complete discussion on the homotopy class and the Novikov covering.)

The following energy estimate will be proved in Section 5. See Proposition 5.4.

Lemma 3.1. *Let $0 \le K < \infty$ and u be any finite energy solution of* (3.3), (3.4). *Then*

$$E_{(J,H,\rho_K)}(u) \le \int u^*\omega + \|H\|. \tag{3.7}$$

We consider the parameterized moduli space

$$\mathcal{M}^{para}(p_-, p_+; B) = \bigcup_{K \in \mathbb{R}_{\ge 0}} \{K\} \times \mathcal{M}^K(p_-, p_+; B)$$

where $\mathcal{M}^K(p_-, p_+; B)$ is the space of solutions in class B to the equation (3.3) satisfying the conditions (3.4) and (3.5) for the parameter $K \in [0, \infty)$. We note that the symplectic area $\int_{\mathbb{R}\times[0,1]} u^*\omega$ is invariant under the homotopy in the space of *smooth* maps with the boundary condition (3.4), which is *fixed*. We shall consider only the triples $(p_-, p_+; B)$ whose associated moduli space $\mathcal{M}^{para}(p_-, p_+; B)$ has virtual dimension 0 or 1.

To study $\mathcal{M}(p_-, p_+; B) = \mathcal{M}^{K=0}(p_-, p_+; B)$, we also need to study the equation

$$\frac{\partial u}{\partial \tau} + J \frac{\partial u}{\partial t} = 0$$

with the same fixed Lagrangian boundary condition but possibly with different asymptotic condition (p'_-, p'_+). The associated moduli space carries the natural $\mathbb{R}$-action and denote by

$$\overline{\mathcal{M}}(p'_-, p'_+; B')$$

the compactification of its quotient by this $\mathbb{R}$-action. We shall however need to consider only those $(p'_-, p'_+; B')$ whose associated moduli space has virtual dimension 0.

First since all the nontrivial holomorphic disk which bounds either $L^{(0)}$ or $L^{(1)}$ have Maslov index ≥ 2 and all the holomorphic spheres have Chern number ≥ 2 we can easily perturb those moduli spaces (of virtual dimension 0 or 1) so that they do not have disk or sphere bubble. We would like to point out that the necessary transversality result on the moduli spaces can be easily achieved in the current context: For $\mathcal{M}^{para}(p_-, p_+)$ we can perturb Hamiltonian term on compact set to make it transversal. For $\overline{\mathcal{M}}(p_-, p_+)$ we can directly check that this moduli space is transversal using the fact that our chosen almost complex structure J is the standard integrable one on $\mathbb{C} \times S^2(1)$.

Now we study the end and the boundary of $\mathcal{M}^{para}(p, p; 0)$ for a $p \in L^{(0)} \cap L^{(1)}$. The space $\mathcal{M}^{para}(p, p; 0)$ is one dimensional. Here $0 \in \Pi(p, p)$ denotes the equivalence class corresponding to the constant map $u \equiv p$. In particular, we have

$$\int u^*\omega = 0. \tag{3.8}$$

We observe that the condition $\psi_H(L^{(0)}) \cap L^{(1)} = \emptyset$ implies the following

Lemma 3.2. *For all sufficiently large $K > 0$, (3.3) has no solution. Namely, we have $\mathcal{M}^K(p, p; 0) = \emptyset$.*

Proof. We refer to the end of p. 901 of [Oh3] for its proof. □

So the boundary of the compactified moduli space

$$\overline{\mathcal{M}}^{para}(q, p; 0)$$

consist of the following three types:

(1) $K = 0$.

(2)
$$\bigcup_q \overline{\mathcal{M}}(p,q;B_1) \times \mathcal{M}^{para}(q,p;B_2),\ B_1 + B_2 = 0. \tag{3.9}$$

(3)
$$\bigcup_q \mathcal{M}^{para}(p,q;B_1) \times \overline{\mathcal{M}}(q,p:B_2),\ B_1 + B_2 = 0. \tag{3.10}$$

The case (1) gives rise to exactly one element. (That is the constant map, $u \equiv p$.) Therefore the sum of the numbers of the boundaries of type (3.9) and of (3.10) must be odd. We will show that this is impossible.

For this purpose, we examine each of the two types in detail. We first consider the case of (3.9). Let

$$(v,(u,K_0)) \in \overline{\mathcal{M}}(p,q;B_1) \times \mathcal{M}^{para}(q,p;B_2).$$

The energy bound (3.7) and (3.5) yield the inequality

$$0 \leq E_{(J,H;\rho_K)}(u) \leq \int u^*\omega + \|H\|$$

for any solution of (3.3) for any K. Therefore from $\|H\| = S - c < S$ and (3.2), we derive

$$\int u^*\omega \geq -\|H\| = -S + c. \tag{3.11}$$

Since we consider an element v of $\overline{\mathcal{M}}(p,q;B_1)$ whose virtual dimension is 0, the element v must be of the product form into

(a) $(v_1, \{pt\})$ with $v_1 : \mathbb{R} \times [0,1] \to \mathbb{C}$ holomorphic

(b) $(\{pt\}, v_2)$ with $v_2 : \mathbb{R} \times [0,1] \to S^2(1)$ holomorphic.

First consider the case (a). We note that there are three bounded connected components of $\mathbb{C} \setminus (S^1(S) \cup S^1_2(S))$. Two of them say D_1, D_2 have small area and the third one D_3 has area $S - \epsilon$ for some $\epsilon > 0$. By choosing a small isometry that we uses in the beginning to define L_1, we may choose $\epsilon \leq \frac{c}{3}$ so that

$$\epsilon + \frac{c}{3} + \|H\| < S - \frac{c}{3}.$$

For this choice of $\varepsilon > 0$, we claim D_3 cannot appear in the compactification of $\mathcal{M}^{para}(p,p;0)$. In fact, if it did, $u \in \mathcal{M}^{para}(q,p;B_2)$ and D_3 would give an element of this compactification that lead to

$$0 = \int u^*\omega + \int v^*\omega \geq -S + c + Area(D_3) \geq -S + c + S - \frac{c}{3} = \frac{2c}{3} > 0,$$

a contradiction.

Note that D_1 and D_2 have the same area $\int_{D_i} \omega = \omega(B_1)$. Therefore the end of $\mathcal{M}^{para}(q,p;0)$ will come in a pair of the form $(v_\pm,(u,K_0))$ contained in

$$\overline{\mathcal{M}}(p,q;B_1) \times \mathcal{M}^{para}(q,p;B_2)$$

so that for each given $(u,K_0) \in \mathcal{M}^{para}(q,p;B_2)$, there is a pair v_-, $v_+ \in \overline{\mathcal{M}}(p,q;B_1)$. Therefore the cardinality of this set must be even.

For the case (b), similarly the end element again comes in a pair of the form $((u,K_0),v_\pm)$ with the the same area $\int v_-^*\omega = \int v_+^*\omega$. (*Here we use the fact that we put the equator on* $S^2(1)$.) Again the cardinality of this set is even.

By the same argument, (3.10) consists of even number of points. This is a contradiction. □

4. Comparison of two Cauchy-Riemann equations and coordinate change

In this section, we explain what the *coordinate change* we called in the introduction means. To describe it precisely, we briefly recall the Novikov covering spaces of the path spaces joining Lagrangian submanifolds, on which the action functional will be defined.

We denote the path space by

$$\Omega(L^{(0)}, L^{(1)}) = \{\ell : [0,1] \to X \mid \ell(0) \in L^{(0)},\ \ell(1) \in L^{(1)}\}.$$

We first recall the action one-form α on $\Omega(L^{(0)}, L^{(1)})$ defined by

$$\alpha(\ell)(\xi) = \int_0^1 \omega(\dot{\ell}(t), \xi(t))\, dt. \tag{4.1}$$

This form is a 'closed' one-form but not 'exact' in general. Due to the presence of the multi-valuedness of the associated action functional we consider the Novikov covering space $\widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a)$ of the connected component $\Omega(L^{(0)}, L^{(1)}; \ell_a)$ containing a chosen base path $\ell_a \in \Omega(L^{(0)}, L^{(1)})$ for each

$$a \in \pi_0(\Omega(L^{(0)}, L^{(1)})),$$

and consider its associated action functional

$$\mathcal{A}_{\ell_a}([\ell, w]) = \int w^*\omega \quad \text{for} \quad [\ell, w] \in \Omega(L^{(0)}, L^{(1)}; \ell_a). \tag{4.2}$$

A simple computation shows that

$$d\mathcal{A}_{\ell_a} = -\pi^*\alpha \tag{4.3}$$

on $\widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a)$. Here $[\ell, w] \in \widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a)$ is an equivalence class of the pair (ℓ, w) of $\ell \in \Omega(L^{(0)}, L^{(1)}; \ell_a)$ and $w : [0,1]^2 \to X$ satisfying

$$w(s,0) \in L^{(0)},\ w(s,1) \in L^{(1)},\ w(0,t) = \ell_a(t),\ w(1,t) = \ell(t)$$

and $\pi : \widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a) \to \Omega(L^{(0)}, L^{(1)})$ is the natural projection given by $[\ell, w] \mapsto \ell$. The equivalence relation is given as follows; $(\ell, w_1) \sim (\ell, w_2)$ if and only if

$$\omega([\overline{w}_1 \# w_2]) = 0 = \mu_{L^{(0)}L^{(1)}}(\overline{w}_1 \# w_2) \tag{4.4}$$

where $\mu_{L^{(0)}L^{(1)}}$ is the Maslov index of the annulus map $\overline{w} \# w' : S^1 \times [0,1] \to X$ with boundary lying on $L^{(0)}$ at $t = 0$ and on $L^{(1)}$ at $t = 1$. We refer to Definition 2.2.4 [FOOO1] for the precise definition of $\widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a)$.

Remark 4.1. In this section and the next, we pick and discuss one connected component of $\Omega(L^{(0)}, L^{(1)})$ and its Novikov covering space without loss of generality. In Section 6, we will consider all connected components to study Floer chain complex which is in fact a direct sum of Floer chain complex for each connected component.

Now for a pair $(L^{(0)}, L^{(1)})$ of compact Lagrangian submanifolds we consider the Hamiltonian deformation $(L^{(0)\prime}, L^{(1)\prime})$ given by

$$L^{(0)\prime} \in \mathfrak{Iso}(L^{(0)}), \quad L^{(1)\prime} \in \mathfrak{Iso}(L^{(1)}).$$

We also consider a family $J^s = \{J_t^s\}_{0\le t\le 1}$ of ω-compatible almost complex structures. We take Hamiltonian isotopies $\phi_{H^{(0)}} = \{\phi^s_{H^{(0)}}\}_{0\le s\le 1}$, $\phi_{H^{(1)}} = \{\phi^s_{H^{(1)}}\}_{0\le s\le 1}$ such that

$$\phi^1_{H^{(0)}}(L^{(0)}) = L^{(0)\prime}, \quad \phi^1_{H^{(1)}}(L^{(1)}) = L^{(1)\prime}. \tag{4.5}$$

For a given pair of Hamiltonian isotopies $\phi_{H^{(i)}}$ for $i = 0, 1$, $J^s = \{J^s_t\}$, and a given smooth function ρ as in Section 2, we consider moving Lagrangian boundary value problem

$$\begin{cases} \frac{\partial u}{\partial \tau} + J^\rho \frac{\partial u}{\partial t} = 0 \\ u(\tau,0) \in \phi^{\rho(\tau)}_{H^{(0)}}(L^{(0)}),\ u(\tau,1) \in \phi^{\rho(\tau)}_{H^{(1)}}(L^{(1)}) \end{cases} \tag{4.6}$$

where $J^\rho(\tau,t) = J_t^{\rho(\tau)}$. Let $[\ell_p, w] \in \widetilde\Omega(L^{(0)}, L^{(1)}; \ell_a)$ and $[\ell_{q'}, w'] \in \widetilde\Omega(L^{(0)\prime}, L^{(1)\prime}; \ell'_a)$. Here we choose

$$\ell'_a(t) := \phi^1_{H^{(1)}}(\phi^{1-t}_{H^{(1)}})^{-1} \circ \phi^1_{H^{(0)}}(\phi^t_{H^{(0)}})^{-1}(\ell_a(t)) \tag{4.7}$$

as the base path of $\Omega(L^{(0)\prime}, L^{(1)\prime})$. See (4.16) and (4.19) below. (We used the notation $\ell_0^{(\psi^{(0)},\psi^{(1)})}$ in [FOOO1].) We denote by

$$\mathcal{M}^\rho((L^{(1)}, \phi^1_{H^{(1)}}), (L^{(0)}, \phi^1_{H^{(0)}}); [\ell_p, w], [\ell_{q'}, w']) \tag{4.8}$$

the set of solutions of (4.6) with

$$[\ell_{q'}, I^\rho_{\phi^1_{H^{(0)}},\phi^1_{H^{(1)}}}(w\#u)] = [\ell_{q'}, w'],$$

where $w\#u : [0,1]\times[0,1] \to X$ is the concatenation in the τ-direction of w and u and

$$I^\rho_{\phi^1_{H^{(0)}},\phi^1_{H^{(1)}}} v(\tau,t) = \left(\phi^1_{H^{(1)}}(\phi^{\rho(\tau)(1-t)}_{H^{(1)}})^{-1} \circ \phi^1_{H^{(0)}}(\phi^{\rho(\tau)t}_{H^{(0)}})^{-1}\right) v(\tau,t).$$

(see p. 308 in [FOOO1]). We note that we do not use the equation (4.6) of *moving* Lagrangian boundary value problem and the moduli space (4.8) themselves in this article, while in Subsection 5.3.2 of [FOOO1] we used them for construction of a filtered A_∞ bimodule homomorphism.

The main goal of Section 4 - Section 6 is to construct a suitable filtered A_∞ bimodule homomorphism, hence a chain map by considering the Hamiltonian perturbed Cauchy-Riemann equation (5.1) different from (4.6), when the pair $(L^{(0)}, L^{(1)})$ is unobstructed in the sense of Lagrangian Floer cohomology theory,

$$C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov}) \to C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\rm nov})$$

in the point of view of filtration changes. See the beginning of Section 6 for a quick review of $C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov})$. The reason why we use the equation (5.1) instead of (4.6) to construct a desired filtered A_∞ bimodule homomorphism is to apply the improved estimates for solutions of (5.1) carried out in Section 5.

First, we consider the particular pairs

$$(L^{(0)}, L^{(1)}), \quad (L^{(0)\prime}, L^{(1)\prime}) = (\phi^1_H(L^{(0)}), L^{(1)})$$

to explain the meaning of the *coordinate change*. These particular pairs correspond to the special case $H^{(1)} \equiv 0$ in the general discussion above.

We would like to compare the *geometric* version of Floer theory and the *dynamical* one. Such a comparison is by now well-known, and the case

$$(L^{(0)}, L^{(1)}), \quad (\phi^1_H(L^{(0)}), L^{(1)}) \tag{4.9}$$

was exploited previously in [Oh2] for the exact Lagrangian submanifolds on the cotangent bundle in relation to the study of spectral invariants. Here we need such a study for general compact Lagrangian submanifolds on general (X,ω).

The geometric version of the Floer complex for $(L^{(0)\prime} = \phi_H^1(L^{(0)}), L^{(1)\prime} = L^{(1)})$ is generated by the intersection points

$$\phi_H^1(L^{(0)}) \cap L^{(1)}$$

and its Floer boundary map is constructed by the moduli space of genuine Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u'}{\partial \tau} + J' \frac{\partial u'}{\partial t} = 0 \\ u'(\tau, 0) \in \phi_H^1(L^{(0)}), \quad u'(\tau,1) \in L^{(1)}. \end{cases} \tag{4.10}$$

Here $J' = J'_t = (\phi_H^1(\phi_H^t)^{-1})_* J_t$. We denote by $\mathcal{M}(L^{(1)}, \phi_H^1(L^{(0)}); J')$ the moduli space of finite energy solutions of this equation. Due to the presence of the multi-valuedness of the associated action functional, we need to consider these equations on the Novikov covering space of some specified connected component

$$\Omega(\phi_H^1(L^{(0)}), L^{(1)}; \ell'_a)$$

with the base path $\ell'_a \in \Omega(\phi_H^1(L^{(0)}), L^{(1)})$, which is given by

$$\ell'_a(t) = \phi_H^1(\phi_H^t)^{-1}(\ell_a(t)), \tag{4.11}$$

and consider the action functional

$$\mathcal{A}_{\ell'_a}([\ell', w']) = \int (w')^* \omega \tag{4.12}$$

where $[\ell', w'] \in \widetilde{\Omega}(\phi_H^1(L^{(0)}), L^{(1)}; \ell'_a)$ and $w' : [0,1]^2 \to X$ is a map satisfying the boundary condition

$$w'(0,t) = \ell'_a(t),\ w'(1,t) = \ell'(t),\ w'(s,0) \in \phi_H^1(L^{(0)}),\ w'(s,1) \in L^{(1)}.$$

On the other hand the dynamical version of the Floer complex is generated by the solutions of Hamilton's equation

$$\dot{x} = X_H(t,x), \quad x(0) \in L^{(0)},\ x(1) \in L^{(1)} \tag{4.13}$$

and its boundary map is constructed by the moduli space of perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0 \\ u(\tau,0) \in L^{(0)}, \quad u(\tau,1) \in L^{(1)}. \end{cases} \tag{4.14}$$

We denote by $\mathcal{M}(L^{(1)}, L^{(0)}; H; J)$ the moduli space of finite energy solutions of this equation. The action functional $\mathcal{A}_{H,\ell_a}$ is defined by

$$\mathcal{A}_{H,\ell_a}([\ell, w]) = \int w^*\omega + \int_0^1 H(t, \ell(t))\, dt \tag{4.15}$$

on $\widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a)$.

These two Floer theories are related by the following transformations of the bijective map

$$\mathfrak{g}^+_{H;0} : \widetilde{\Omega}(\phi_H^1(L^{(0)}), L^{(1)}; \ell'_a) \to \widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a); \quad [\ell', w'] \mapsto [\ell, w]$$

given by the assignment

$$\ell(t) = \phi_H^t(\phi_H^1)^{-1}(\ell'(t)), \quad w(s,t) = \phi_H^t(\phi_H^1)^{-1}(w'(s,t)). \tag{4.16}$$

This provides a bijective correspondence of the critical points

$$\mathrm{Crit}\,\mathcal{A}_{\ell'_a} \longleftrightarrow \mathrm{Crit}\,\mathcal{A}_{H,\ell_a}; \quad [p',w'] \mapsto [z^H_{p'},w] \tag{4.17}$$

with $p' \in \phi^1_H(L^{(0)}) \cap L^{(1)}$, $z^H_{p'}(t) := \phi^t_H(\phi^1_H)^{-1}(p')$ and $w = \phi^t_H(\phi^1_H)^{-1}(w'(s,t))$, and of the moduli spaces

$$\mathcal{M}(L^{(1)},\phi^1_H(L^{(0)});J') \mapsto \mathcal{M}(L^{(1)},L^{(0)};H;J)$$

with $J_t = (\phi^t_H(\phi^1_H)^{-1})_* J'_t$ where the map is defined by

$$u(\tau,t) = \phi^t_H(\phi^1_H)^{-1}(u'(\tau,t)).$$

The map $\mathfrak{g}^+_{H;0}$ also preserves the action up to a constant in that

Lemma 4.2. *Denote*

$$c(H;\ell_a) := \int_0^1 H(t,\ell_a(t))\,dt$$

which is a constant depending only on H and the base path ℓ_a of the connected component $\Omega(L^{(0)},L^{(1)};\ell_a)$. Then

$$\mathcal{A}_{H,\ell_a} \circ \mathfrak{g}^+_{H;0}([\ell',w']) = \mathcal{A}_{\ell'_a}([\ell',w']) + c(H;\ell_a) \tag{4.18}$$

on $\widetilde\Omega(L^{(0)},L^{(1)};\ell_a)$.

Remark 4.3. Since we normalized Hamiltonians so that $\int_X H_t\omega^n = 0$ for each t in Section 2, we can take ℓ_a for each connected component of $\Omega(L^{(0)},L^{(1)})$ in such a way that $c(H;\ell_a) = \int_0^1 H(t,\ell_a(t))dt = 0$. It is not essential to choose ℓ_a in a way as above. In fact, if we take a based path so that $c(H;\ell_a) \neq 0$, it is enough to include an extra term $c(H;\ell_a)$ in the energy estimate on $\Omega(L^{(0)},L^{(1)};\ell_a)$ when we apply the coordinate change $\mathfrak{g}^+_{H;0}$ or its inverse. Since we will consider all connected components of $\Omega(L^{(0)},L^{(1)})$ in Section 6 (see Remark 4.1), we have to add the constant $c(H;\ell_a)$ for each connected component $\Omega(L^{(0)},L^{(1)};\ell_a)$. Thus, to avoid heavy notation, we simply choose ℓ_a so that $c(H;\ell_a) = 0$ for each connected component $\Omega(L^{(0)},L^{(1)};\ell_a)$.

Proof. The proof is by a direct calculation. Let $[\ell',w'] \in \widetilde\Omega(\phi^1_H(L^{(0)}),L^{(1)};\ell'_a)$. Then

$$\mathcal{A}_{H;\ell_a}(\mathfrak{g}^+_{H;0}([\ell',w'])) = \int w^*\omega + \int_0^1 H(t,\ell(t))\,dt$$

and

$$w^*\omega = \omega\left(\frac{\partial w}{\partial s},\frac{\partial w}{\partial t}\right) ds \wedge dt.$$

We compute

$$\begin{aligned}
\frac{\partial w}{\partial s} &= d(\phi^t_H(\phi^1_H)^{-1})\left(\frac{\partial w'}{\partial s}\right)\\
\frac{\partial w}{\partial t} &= d(\phi^t_H(\phi^1_H)^{-1})\left(\frac{\partial w'}{\partial t}\right) + X_{H_t}(w(s,t)).
\end{aligned}$$

Substituting this into the above, we obtain

$$
\begin{aligned}
\int w^*\omega &= \int_0^1\int_0^1 \omega\left(\frac{\partial w}{\partial s},\frac{\partial w}{\partial t}\right) ds\,dt \\
&= \int_0^1\int_0^1 \omega\left(\frac{\partial w'}{\partial s},\frac{\partial w'}{\partial t}\right) ds\,dt \\
&\qquad + \int_0^1\int_0^1 \omega\left(d\left(\phi_H^t(\phi_H^1)^{-1}\right)\left(\frac{\partial w'}{\partial s}\right), X_{H_t}(w(s,t))\right) ds\,dt \\
&= \int (w')^*\omega - \int_0^1\int_0^1 dH_t(w(s,t))\left(d(\phi_H^t(\phi_H^1)^{-1})\frac{\partial w'}{\partial s}\right) ds\,dt \\
&= \int (w')^*\omega - \int_0^1\int_0^1 \frac{\partial}{\partial s}H_t(w(s,t))\,ds\,dt \\
&= \int (w')^*\omega - \int_0^1 H_t(w(1,t))\,dt + \int_0^1 H_t(w(0,t))\,dt.
\end{aligned}
$$

Substituting this into the above definition of $\mathcal{A}_{H;\ell_a}(\mathfrak{g}^+_{H;0}([\ell',w']))$, the proof is finished. □

We denote by $\mathfrak{g}^-_{H;0}$ the inverse $\mathfrak{g}^-_{H;0} = (\mathfrak{g}^+_{H;0})^{-1}$. The outcome of the above discussion is that the two associated Floer cohomologies are isomorphic to each other.

So far we have moved the first argument $L^{(0)}$ in the pair $(L^{(0)}, L^{(1)})$. We can also move the second argument $L^{(1)}$ instead. In that case, we define the coordinate change transformation by

$$\mathfrak{g}^+_{\widetilde{H};1} : \widetilde{\Omega}(L^{(0)}, \phi_H^1(L^{(1)}); \ell'_a) \to \widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a); \quad [\ell', w'] \mapsto [\ell, w]$$

given by the assignment

$$\ell(t) = \phi_H^{1-t}(\phi_H^1)^{-1}(\ell'(t)), \quad w(s,t) = \phi_H^{1-t}(\phi_H^1)^{-1}(w'(s,t)) \tag{4.19}$$

where $\widetilde{H}(t,x) := -H(1-t,x)$ is the Hamiltonian generating the latter Hamiltonian path $t \mapsto \phi_H^{1-t}(\phi_H^1)^{-1}$. This provides a bijective correspondence

$$\operatorname{Crit}\mathcal{A}_{\ell'_a} \longleftrightarrow \operatorname{Crit}\mathcal{A}_{\widetilde{H},\ell_a}$$

and the moduli spaces

$$\mathcal{M}(\phi_H^1(L^{(1)}), L^{(0)}; J') \mapsto \mathcal{M}(\widetilde{H}; L^{(1)}, L^{(0)}; J)$$

with $\widetilde{J}_t = (\phi_H^{1-t}(\phi_H^1)^{-1})_* J'_t$. Here $\mathcal{M}(\widetilde{H}; L^{(1)}, L^{(0)}; \widetilde{J})$ is the moduli space of solutions of

$$
\begin{cases}
\frac{\partial u}{\partial \tau} + \widetilde{J}\left(\frac{\partial u}{\partial t} - X_{\widetilde{H}}(u)\right) = 0 \\
u(\tau,0) \in L^{(0)},\ u(\tau,1) \in L^{(1)}.
\end{cases}
\tag{4.20}
$$

The action functional $\mathcal{A}_{\widetilde{H},\ell_a}$ is given by

$$\mathcal{A}_{\widetilde{H},\ell_a}([\ell,w]) = \int (\widetilde{w})^*\omega + \int_0^1 \widetilde{H}(t,\ell(t))\,dt. \tag{4.21}$$

The explicit formula of the latter correspondence is given by

$$u(\tau,t)) = \phi_H^{1-t}(\phi_H^1)^{-1}(u'(\tau,t)).$$

Again the following can be proved by a similar computations used to prove Lemma 4.2 whose proof is left to the readers.

Lemma 4.4. *We have*

$$\mathcal{A}_{\widetilde{H},\ell_a} \circ \mathfrak{g}^+_{\widetilde{H};1}([\ell', w']) = \mathcal{A}_{\ell'_a}([\ell', w']) + c(\widetilde{H}; \ell_a) \tag{4.22}$$

where

$$c(\widetilde{H}; \ell_a) := \int_0^1 \widetilde{H}(t, \ell_a(t))\, dt$$

is a constant depending only on H and the base path ℓ_a.

We denote by $\mathfrak{g}^-_{\widetilde{H};1}$ the inverse of $\mathfrak{g}^+_{\widetilde{H};1}$.

5. Improved energy estimate

First we consider the case of varying the first argument $L^{(0)}$ of the pair $(L^{(0)}, L^{(1)})$ $(L^{(0)\prime}, L^{(1)\prime}) = (\phi^1_H(L^{(0)}), L^{(1)})$. In this case, as far as the study of the optimal filtration change is concerned, employing the moduli space with moving Lagrangian boundary is not the best one. We will show that employing the standard *perturbed* Cauchy-Riemann equation by Hamiltonian vector fields with *fixed* Lagrangian boundaries, which intertwines the geometric version and the dynamical version of the Floer complex, gives a stronger energy estimate which gives rise to the optimal change of filtration.

Let ρ be one of smooth functions on $\mathbb{R}$ of the three types introduced in Section 2. See (2.4) and (2.5). Consider the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - \rho(\tau) X_H(u)\right) = 0 \\ u(\tau, 0) \in L^{(0)},\ u(\tau, 1) \in L^{(1)} \end{cases} \tag{5.1}$$

with the finite energy $E_{(J,H,\rho)}(u) < \infty$. The following a priori energy bound is a key ingredient in relation to the lower bound of displacement energy. This kind of optimal estimate is originally due to Chekanov [Che], which is the key calculation that relates the energy and the Hofer norm in an optimal way. For completeness' sake, we include its proof which is a slight variation of the calculation carried out in p. 901 [Oh3]. It is useful to decompose $\|H\|$ into two parts

$$E^-(H) = \int_0^1 -\min\ H_t\, dt, \quad E^+(H) = \int_0^1 \max\ H_t\, dt,$$

which are so called the negative and positive part of the Hofer norm $\|H\|$.

Lemma 5.1. *Let $\rho = \rho_+$ as in* (2.4)*. Let u be any finite energy solution of* (5.1)*. Then we have*

$$\begin{aligned} E_{(J,H,\rho)}(u) &= \int u^*\omega + \int_0^1 H(t, u(\infty, t))\, dt \\ &\quad - \int_{-\infty}^{\infty} \rho'(\tau) \int_0^1 (H_t \circ u)\, dt\, d\tau. \end{aligned} \tag{5.2}$$

Proof. The proof will be carried out by an explicit calculation. We compute

$$
\begin{aligned}
E_{(J,H,\rho)}(u) &= \int_{-\infty}^{\infty}\int_0^1 \left|\frac{\partial u}{\partial \tau}\right|_J^2 dt\,d\tau = \int_{-\infty}^{\infty}\int_0^1 \omega\left(\frac{\partial u}{\partial \tau}, J\frac{\partial u}{\partial \tau}\right) dt\,d\tau \\
&= \int_{-\infty}^{\infty}\int_0^1 \omega\left(\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial t} - \rho(\tau)X_{H_t}(u)\right) dt\,d\tau \\
&= \int_{-\infty}^{\infty}\int_0^1 \omega\left(\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial t}\right) dt\,d\tau - \int_{-\infty}^{\infty}\rho(\tau)\int_0^1 \omega\left(\frac{\partial u}{\partial \tau}, X_{H_t(u)}\right) dt\,d\tau \\
&= \int u^*\omega - \int_{-\infty}^{\infty}\rho(\tau)\int_0^1 \left(-dH_t(u)\frac{\partial u}{\partial \tau}\right) dt\,d\tau \\
&= \int u^*\omega + \int_{-\infty}^{\infty}\rho(\tau)\int_0^1 \frac{\partial}{\partial \tau}(H_t\circ u)\,dt\,d\tau \\
&= \int u^*\omega + \int_0^1 H(t,u(\infty,t))\,dt - \int_{-\infty}^{\infty}\rho'(\tau)\int_0^1 (H_t\circ u)\,dt\,d\tau.
\end{aligned}
$$

Here at the last equality, we do integration by parts and use the fact $\rho(\infty) = 1$, $\rho(-\infty) = 0$. This finishes the proof of (5.2). □

This lemma gives rise to the following key formula of the action difference

$$\mathcal{A}_{H,\ell_a}(u(\infty)) - \mathcal{A}_{\ell_a}(u(-\infty)).$$

Proposition 5.2. *Let $p \in L^{(0)} \cap L^{(1)}$ and $q' \in \phi_H^1(L^{(0)}) \cap L^{(1)}$. Denote by $z_{q'}^H \in \Omega(L^{(0)}, L^{(1)}; \ell_a)$ the Hamiltonian trajectory defined by*

$$z_{q'}^H(t) = \phi_H^t(\phi_H^{-1}(q'))$$

and consider $[\ell_p, w] \in \mathrm{Crit}\,\mathcal{A}_{\ell_a}$, $[z_{q'}^H, w'] \in \mathrm{Crit}\,\mathcal{A}_{H,\ell_a}$. Suppose that u is any finite energy solution of (5.1) with $\rho = \rho_+$ as in (2.4) *satisfying the asymptotic condition and homotopy condition*

$$u(-\infty) = \ell_p, \quad u(\infty) = z_{q'}^H, \quad w\#u \sim w'. \tag{5.3}$$

Then we have

$$\mathcal{A}_{H,\ell_a}([z_{q'}^H, w']) - \mathcal{A}_{\ell_a}([w, \ell_p]) = E_{(J,H,\rho)}(u) + \int_{-\infty}^{\infty}\int_0^1 \rho'(\tau)H(t,u(\tau,t))\,dt\,d\tau. \tag{5.4}$$

Proof. By (5.3), we obtain

$$\int u^*\omega = \int (w')^*\omega - \int w^*\omega.$$

By substituting this into (5.2) and rearranging the resulting formula, we obtain (5.4) from the definitions (4.12) of $\mathcal{A}_{\ell_a}$ and (4.15) of $\mathcal{A}_{H,\ell_a}$. □

An immediate corollary is

Corollary 5.3. *Suppose that there is a solution u of* (5.1) *for $\rho = \rho_+$ as in Proposition 5.2. Then we have*

$$\mathcal{A}_{H,\ell_a}([z_{q'}^H, w']) - \mathcal{A}_{\ell_a}([\ell_p, w]) \geq \int_0^1 \min H_t\,dt = -E^-(H). \tag{5.5}$$

Similarly if there is a solution u of (5.1) *for $\rho = \rho_- = 1 - \rho_+$,*

$$\mathcal{A}_{\ell_a}([\ell_q, w]) - \mathcal{A}_{H,\ell_a}([z^H_{p'}, w']) \geq \int_0^1 -\max H_t\, dt = -E^+(H). \tag{5.6}$$

Next let us concatenate the two equation (5.1) for ρ_+ as in (2.5) and $\rho_- = 1 - \rho_+$ by considering one-parameter family of elongation function of the type

$$\rho_K = \begin{cases} \rho_+(\cdot + K) & \text{for } \tau \leq -K+1 \\ \rho_-(\cdot - K + 1) & \text{for } \tau \geq K - 1 \\ 1 & \text{for } |\tau| \leq K-1 \end{cases} \tag{5.7}$$

for $1 \leq K \leq \infty$ and further deforming $\rho_{K=1}$ further down to $\rho_{K=0} \equiv 0$.

Proposition 5.4. *Let u be a finite energy solution for* (5.1) *of the elongation ρ_K with asymptotic condition*

$$u(-\infty) = \ell_p,\ u(\infty) = \ell_q,\ w_-\# u \sim w_+.$$

Then we have

$$\begin{aligned} \mathcal{A}_{\ell_a}([\ell_q, w_+]) - \mathcal{A}_{\ell_a}([\ell_p, w_-]) \quad &\geq \quad -\big(E^-(H) + E^+(H)\big) = -\|H\| && (5.8)\\ E_{(J,H;\rho_K)}(u) \quad &\leq \quad \int u^*\omega + \|H\|. && (5.9)\end{aligned}$$

So far in this section, we have moved the first argument $L^{(0)}$ in the pair $(L^{(0)}, L^{(1)})$. When we move the second argument $L^{(1)}$ instead, the only difference occurring in the above discussion will be the interchange

$$-\min H \longleftrightarrow \max H.$$

Now we move $L^{(0)}$ and $L^{(1)}$ by Hamiltonian isotopies $\phi^t_{H^{(0)}}$ and $\phi^t_{H^{(1)}}$, respectively.

$$(L^{(0)}, L^{(1)}) \mapsto (L^{(0)\prime} = \phi^1_{H^{(0)}}(L^{(0)}), L^{(1)\prime} = \phi^1_{H^{(1)}}(L^{(1)})).$$

Then we have the following bijection

$$\mathfrak{g}^+_{H^{(0)},H^{(1)}} : (\ell'', w'') \in \widetilde{\Omega}(L^{(0)\prime}, L^{(1)\prime}) \mapsto (\ell, w) \in \widetilde{\Omega}(L^{(0)}, L^{(1)}),$$

where

$$\ell(t) = \phi^{1-t}_{H^{(1)}} \circ (\phi^1_{H^{(1)}})^{-1} \circ \phi^t_{H^{(0)}} \circ (\phi^1_{H^{(0)}})^{-1}(\ell''(t))$$

and

$$w(s,t) = \phi^{1-t}_{H^{(1)}} \circ (\phi^1_{H^{(1)}})^{-1} \circ \phi^t_{H^{(0)}} \circ (\phi^1_{H^{(0)}})^{-1}(w''(s,t)).$$

We write $\mathfrak{g}^-_{H^{(0)},H^{(1)}} = (\mathfrak{g}^+_{H^{(0)},H^{(1)}})^{-1}$. By an abuse of notation, we also denote by $\mathfrak{g}^\pm_{H^{(0)},H^{(1)}}$ the bijection between the path spaces $\Omega(L^{(0)}, L^{(1)})$ and $\Omega(L^{(0)\prime}, L^{(1)\prime})$. Then we obtain the following improved energy estimate. Here we take the base path ℓ_a in such a way that

$$c(\widehat{H}; \ell_a) = \int_0^1 \widehat{H}(t, \ell_a(t))dt = 0 \tag{5.10}$$

as in Remark 4.3. Here $\widehat{H}$ is the normalized Hamiltonian generating

$$\phi^{1-t}_{H^{(1)}} \circ (\phi^1_{H^{(1)}})^{-1} \circ \phi^t_{H^{(0)}}.$$

The Hamiltonian $\widehat{H}$ is explitly written as

$$\widehat{H}(t,x) = -H^{(1)}(1-t, x) + H^{(0)}(t, (\phi^{1-t}_{H^{(1)}} \circ (\phi^1_{H^{(1)}})^{-1})^{-1}(x)). \tag{5.11}$$

For $v : [0,1] \times [0,1] \to X$, we put

$$(\Phi_{H^{(0)},H^{(1)}} v)(s,t) = \phi^1_{H^{(0)}} \circ (\phi^t_{H^{(0)}})^{-1} \circ \phi^1_{H^{(1)}} \circ (\phi^{1-t}_{H^{(1)}})^{-1} v(s,t).$$

By the expression (5.11) of $\widehat{H}$, we find that

$$E^-(\widehat{H}) \le E^-(H^{(0)}) + E^+(H^{(1)}), \quad E^+(\widehat{H}) \le E^+(H^{(0)}) + E^-(H^{(1)}). \tag{5.12}$$

Recall that we have chosen ℓ_a such that (5.10) is satisfied and put $\ell''_a = \mathfrak{g}^-_{H^{(0)},H^{(1)}}(\ell_a)$.

Proposition 5.5. *Let $(L^{(0)}, L^{(1)})$ be a pair of compact Lagrangian submanifolds and $(L^{(0)\prime}, L^{(1)\prime})$ another pair with*

$$L^{(0)\prime} = \phi^1_{H^{(0)}}(L^{(0)}), \quad L^{(1)\prime} = \phi^1_{H^{(1)}}(L^{(1)})$$

and let $H^{(0)}$, $H^{(1)}$ be the normalized Hamiltonians generating $\phi^1_{H^{(0)}}$ and $\phi^1_{H^{(1)}}$ respectively. Consider a pair $[\ell_p, w] \in \widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a)$ and $[\ell_{q''}, w''] \in \widetilde{\Omega}(L^{(0)\prime}, L^{(1)\prime}; \ell''_a)$ for which there exists a solution u of (5.1) with $\rho = \rho_+$ as in (2.4) such that

$$\lim_{\tau\to-\infty} u(\tau,\cdot) = \ell_p, \ \lim_{\tau\to+\infty} u(\tau,\cdot) = \mathfrak{g}^+_{H^{(0)},H^{(1)}}(\ell_{q''}), \ \Phi_{H^{(0)},H^{(1)}}(w\#u) \sim w''.$$

Then we have

$$\mathcal{A}_{\ell''_a}([\ell_{q''}, w'']) - \mathcal{A}_{\ell_a}([\ell_p, w]) \ge -(E^-(H^{(0)}) + E^+(H^{(1)})). \tag{5.13}$$

Similarly, let $[\ell_{p''}, w''] \in \widetilde{\Omega}(L^{(0)\prime}, L^{(1)\prime}; \ell''_a)$ and $[\ell_q, w] \in \widetilde{\Omega}(L^{(0)}, L^{(1)}; \ell_a)$. If there exists a solution u of (5.1) with $\rho = \rho_- = 1 - \rho_+$ such that

$$\lim_{\tau\to-\infty} u(\tau,\cdot) = \mathfrak{g}^+_{H^{(0)},H^{(1)}}(\ell_{p''}), \ \lim_{\tau\to+\infty} u(\tau,\cdot) = \ell_q, \ \Phi^{-1}_{H^{(0)},H^{(1)}}(w'')\#u \sim w,$$

we have

$$\mathcal{A}_{\ell_a}([\ell_q, w]) - \mathcal{A}_{\ell''_a}([\ell_{p''}, w'']) \ge -(E^+(H^{(0)}) + E^-(H^{(1)})). \tag{5.14}$$

The following proposition is parallel to Proposition 5.4

Proposition 5.6. *Let $(L^{(0)}, L^{(1)})$ be a pair of compact Lagrangian submanifolds. If there exists a solution u of (5.1) with $\rho = \rho_K$ in (5.7) satisfying*

$$\lim_{\tau\to-\infty} u(\tau,\cdot) = \ell_p, \ \lim_{\tau\to+\infty} u(\tau,\cdot) = \ell_q, \ w_-\#u \sim w_+.$$

Then we have

$$\mathcal{A}_{\ell_a}([\ell_q, w_+]) - \mathcal{A}_{\ell_a}([\ell_p, w_-]) \ge -\left(\|H^{(0)}\| + \|H^{(1)}\|\right) \tag{5.15}$$

and

$$E_{(J,\widehat{H},\rho_K)}(u) \le \int u^*\omega + \|H^{(0)}\| + \|H^{(1)}\|. \tag{5.16}$$

6. Corrected proofs of Theorem J and Theorem 6.1.25 [FOOO1]

To keep the statement of Theorem J [FOOO1] as it is, we need to modify construction of the chain map used in the proof of Theorem 6.1.25 [FOOO1].

In the rest of the paper, we assume that a Lagrangian submanifold is closed and relatively spin and a pair of Lagrangian submanifolds is relatively spin (Definition 1.6 [FOOO1]) unless otherwise noted. In this discussion we use the $\mathbb{C}$-coefficients as in [FOOO2, FOOO3] but one can also use the $\mathbb{Q}$-coefficients as in [FOOO1].

We first recall the definition of the universal Novikov ring $\Lambda_{\rm nov}$ used in [FOOO1]. An element of $\Lambda_{\rm nov}$ is a formal sum

$$\sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i}$$

with $a_i \in \mathbb{C}$, $\lambda_i \in \mathbb{R}$, $\mu_i \in \mathbb{Z}$ such that $\lambda_i \le \lambda_{i+1}$ and $\lim_{i\to\infty} \lambda_i = \infty$, unless it is a finite sum. Here T and e are formal parameters. We define a valuation $\mathfrak{v}_T : \Lambda_{\rm nov} \to \mathbb{R}$ by

$$\mathfrak{v}_T\left(\sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i}\right) = \lambda_1.$$

This induces a natural $\mathbb{R}$-filtration on $\Lambda_{\rm nov}$ which in turn induces a non-Archimedean topology thereon. Then we define $\Lambda_{0,\rm nov}$ to be the subring of $\Lambda_{\rm nov}$ consisting of $\sum a_i T^{\lambda_i} e^{\mu_i}$ with $\mathfrak{v}_T\left(\sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i}\right) \ge 0$ and $\Lambda_{0,\rm nov}^+$ by the subring with $\mathfrak{v}_T > 0$.

We define $C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov})$ as the $\Lambda_{\rm nov}$-module generated by Crit $\mathcal{A}_{\ell_a}$, $a \in \pi_0(\Omega(L^{(0)}, L^{(1)}))$ modulo the equivalence relation $\sim$ given in (4.4). The filtration $\{F^\lambda\}$ on $C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov})$ is given by the action functional $\mathcal{A}_{\ell_a}$. See p.127 in [FOOO1]. We can regard $C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov})$ as a free $\Lambda_{\rm nov}$-module generated by $L^{(0)} \cap L^{(1)}$ provided $L^{(0)}$ and $L^{(1)}$ intersect transversally. In such a situation, we can identify $F^0C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov})$ and the free $\Lambda_{0,\rm nov}$-module generated by $L^{(0)} \cap L^{(1)}$. We defined a filtered A_∞-bimodule structure on $C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov})$ in Theorem 3.7.21 in [FOOO1] (see also Definition 3.7.41 in [FOOO1]). By extending the coefficient ring to $\Lambda_{\rm nov}$, we also have a filtered A_∞-bimodule structure on $C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov})$. This construction does not rely on the choice of the base paths ℓ_a. However, when we construct a filtered A_∞-bimodule homomorphism $C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov}) \to C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\rm nov})$, we use the base baths ℓ_a and ℓ'_a. As we will see, the improved estimate in Section 5 is used to control the filtration change under the filtered A_∞-bimodule homomorphism.

6.1. **Statement of Theorem J** [FOOO1]**.** In [FOOO1], we associate a set $\mathcal{M}_{\rm weak,def}(L)$ for each relatively spin Lagrangian submanifold L of (X, ω) and the maps

$$\pi_{\rm amb} : \mathcal{M}_{\rm weak,def}(L) \to H^2(X; \Lambda_{0,\rm nov}^+), \quad \mathfrak{PO} : \mathcal{M}_{\rm weak,def}(L) \to \Lambda_{0,\rm nov}^+$$

such that the Floer cohomology $HF((L, \mathbf{b}_1), (L, \mathbf{b}_0); \Lambda_{0,\rm nov})$ can be defined whenever the following condition holds: $\mathcal{M}_{\rm weak,def}(L) \ne \emptyset$ and

$$\pi_{\rm amb}(\mathbf{b}_1) = \pi_{\rm amb}(\mathbf{b}_0), \quad \mathfrak{PO}(\mathbf{b}_1) = \mathfrak{PO}(\mathbf{b}_0).$$

See Theorem B [FOOO1]. When this condition is satisfied, we say L is *weakly unobstructed after bulk deformation*. We set

$$\mathcal{M}_{\rm weak}(L) = \pi_{\rm amb}^{-1}(0), \quad \mathcal{M}(L) = \pi_{\rm amb}^{-1}(0) \cap \mathfrak{PO}^{-1}(0),$$

whose elements are called *weak bounding cochain* (weak Maurer-Cartan element), *bounding cochain* (Maurer-Cartan element), respectively. See Section 3.6, especially Definition 3.6.4 and Definition 3.6.29 [FOOO1] for the precise definitions of bounding cochain and weak bounding cochain. More generally, for a relative spin pair $(L^{(1)}, L^{(0)})$ of Lagrangian submanifolds and

$$\begin{aligned}(\mathbf{b}_1, \mathbf{b}_0) \quad &\in \{(\mathbf{b}_1, \mathbf{b}_0) \mid \pi_{\rm amb}(\mathbf{b}_1) = \pi_{\rm amb}(\mathbf{b}_0),\ \mathfrak{PO}(\mathbf{b}_1) = \mathfrak{PO}(\mathbf{b}_0)\} \\ &=: \mathcal{M}_{\rm weak,def}(L^{(1)}) \times_{(\pi_{\rm amb}, \mathfrak{PO})} \mathcal{M}_{\rm weak,def}(L^{(0)}), \qquad (6.1)\end{aligned}$$

we can define the Floer cohomology $HF((L^{(1)},\mathbf{b}_1),(L^{(0)},\mathbf{b}_0);\Lambda_{0,\mathrm{nov}})$ over $\Lambda_{0,\mathrm{nov}}$. By Theorem 6.1.20 [FOOO1], it is isomorphic to

$$\Lambda_{0,\mathrm{nov}}^{\oplus a} \oplus \bigoplus_{i=1}^{b} (\Lambda_{0,\mathrm{nov}}/T^{\lambda_i}\Lambda_{0,\mathrm{nov}}) \tag{6.2}$$

for some non negative integer a and positive real numbers λ_i $(i = 1,\dots,b)$. We call a the *Betti number* and λ_i the *torsion exponent* of the Floer cohomology. We note that $HF((L^{(1)},\mathbf{b}_1),(L^{(0)},\mathbf{b}_0);\Lambda_{0,\mathrm{nov}})$ is not invariant under the Hamiltonian isotopy. However, it is proved in [FOOO1] (see Theorem G (G.4)) that the Floer cohomology

$$HF((L^{(1)},\mathbf{b}_1),(L^{(0)},\mathbf{b}_0);\Lambda_{\mathrm{nov}})$$

with Λ_{nov} its coefficients is invariant under the Hamiltonian isotopy and satisfies

$$HF((L^{(1)},\mathbf{b}_1),(L^{(0)},\mathbf{b}_0);\Lambda_{\mathrm{nov}}) \cong \Lambda_{\mathrm{nov}}^{\oplus a}. \tag{6.3}$$

In particular, when $a \neq 0$, $L^{(0)}$, $L^{(1)}$ can not be displaced from each other. On the other hand, when $a = 0$, there is no obvious obstruction to the displacement. In this case, the torsion part of $HF((L^{(1)},\mathbf{b}_1),(L^{(0)},\mathbf{b}_0);\Lambda_{0,\mathrm{nov}})$ provides some information on the Hofer distance and the displacement energy of the two.

Now, under the above brief review of Lagrangian Floer theory for a weakly unobstructed Lagrangian submanifold after bulk deformation, we can state Theorem J of [FOOO1].

Theorem 6.1 (Theorem J [FOOO1])**.** *Let $(L^{(0)},L^{(1)})$ be a relatively spin pair of Lagrangian submanifolds of X and $L^{(1)},L^{(0)}$ weakly unobstructed after bulk deformations. Let $(\mathbf{b}_1,\mathbf{b}_0) \in \mathcal{M}_{\mathrm{weak,def}}(L^{(1)}) \times_{\pi_{\mathrm{amb}},\mathfrak{PD}} \mathcal{M}_{\mathrm{weak,def}}(L^{(0)})$ as in* (6.1) *and $\psi : X \to X$ a Hamiltonian diffeomorphism. Assume that $\psi(L^{(1)})$ is transversal to $L^{(0)}$ and denote*

$$b(\|\psi\|) = \#\{i \mid \lambda_i \geq \|\psi\|\},$$

where λ_i are the torsion exponents as in (6.2) *and $\|\phi\|$ is the Hofer norm defined by* (2.2)*. Then we have*

$$\#(\psi(L^{(1)}) \cap L^{(0)}) \geq a + 2b(\|\psi\|). \tag{6.4}$$

Theorem 6.1 follows from the following Theorem 6.1.25 of [FOOO1] (see Subsection 6.5.3 [FOOO1]). The proof of Theorem 6.1.25 contained an error which we now fix.

We recall that a symplectic diffeomorphism $\psi : (X,L) \to (X,L')$ induces a bijection

$$\psi_* : \mathcal{M}_{\mathrm{weak,def}}(L) \to \mathcal{M}_{\mathrm{weak,def}}(L')$$

which is compatible with the maps π_{amb} and $\mathfrak{PD}$. See Theorem B (B.3) [FOOO1].

Theorem 6.2 (Theorem 6.1.25 [FOOO1])**.** *Let $(L^{(0)},L^{(1)})$ and $(\mathbf{b}_1,\mathbf{b}_0)$ be as in Theorem 6.1, and $\psi^{(i)} : X \to X$ $(i = 0,1)$ Hamiltonian diffeomorphisms. Put $L^{(i)\prime} = \psi^{(i)}(L^{(i)})$. Let $\lambda_{\downarrow,i}$, $i = 1,\dots,b$ and $\lambda'_{\downarrow,i}$ $i = 1,\dots,b'$ be the torsion exponents of the Floer cohomology*

$$HF((L^{(1)},\mathbf{b}_1),(L^{(0)},\mathbf{b}_0);\Lambda_{0,\mathrm{nov}}),\ \ HF((L^{(1)\prime},\psi^{(1)}_*\mathbf{b}_1),(L^{(0)\prime},\psi^{(0)}_*\mathbf{b}_0);\Lambda_{0,\mathrm{nov}})$$

respectively. We order them so that $\lambda_{\downarrow,i} \geq \lambda_{\downarrow,i+1}$ and $\lambda'_{\downarrow,i} \geq \lambda'_{\downarrow,i+1}$. Denote

$$\nu_0 = \mathrm{dist}(L^{(0)},L^{(0)\prime}) + \mathrm{dist}(L^{(1)},L^{(1)\prime}). \tag{6.5}$$

Then if $\lambda_{\downarrow,i} > \nu_0$, then $i \le b'$, and if $\lambda_{\downarrow,i} > \nu_0$ and $\lambda'_{\downarrow,i} > \nu_0$, then we have

$$| \lambda_{\downarrow,i} - \lambda'_{\downarrow,i} | \le \nu_0. \tag{6.6}$$

In particular, $\lambda_{\downarrow,i}$ is continuous for each i as long as $\lambda_{\downarrow,i} > 0$.

Remark 6.3. Let $\lambda \in \mathbb{R}$ such that $\lambda > 2\|H\|$. In the statement (6.5.30) in p. 391 [FOOO1], we obtained the chain maps

$$\phi : T^{\lambda} C(L^{(1)}, L^{(0)}; \Lambda_{0,\mathrm{nov}}) \to T^{\lambda - \|H\|} C(L^{(1)\prime}, L^{(0)\prime}); \Lambda_{0,\mathrm{nov}})) \tag{6.7}$$

$$\phi' : T^{\lambda - \|H\|} C(L^{(1)}, L^{(0)}; \Lambda_{0,\mathrm{nov}}) \to T^{\lambda - 2\|H\|} C(L^{(1)}, L^{(0)}; \Lambda_{0,\mathrm{nov}}). \tag{6.8}$$

The above mentioned error lies in the fact that the composition of (6.7) and (6.8) only chain homotopy equivalent to the inclusion

$$\mathfrak{i} : T^{\lambda} C(L^{(1)}, L^{(0)}; \Lambda_{0,\mathrm{nov}}) \longrightarrow T^{\lambda - 2\|H\|} C(L^{(1)}, L^{(0)}; \Lambda_{0,\mathrm{nov}})$$

if we use the original energy estimate given in Proposition 5.3.20 (Proposition 5.3.45) [FOOO1]. Therefore we need to replace the rest of the proof by the following argument which uses the construction of an optimal chain maps combining the coordinate transformations explained in the previous sections and the improved energy estimate.

6.2. **Proof of Theorem 6.1.25** [FOOO1]**.** In this subsection we prove Theorem 6.1.25 [FOOO1].

Consider the pair $(\psi^{(0)}, \psi^{(1)})$ of Hamiltonian diffeomorphisms. As in [FOOO1], to simplify the notation, we restrict ourselves to the case of a transverse pair $(L^{(0)}, L^{(1)})$ where both $L^{(i)}$ are unobstructed, i.e., $\mathcal{M}(L^{(i)}) \neq \emptyset$. Then using bounding cochains $b_i \in \mathcal{M}(L^{(i)})$, we can define the coboundary operator δ_{b_1,b_0} on the filtered A_∞ bimodule $C(L^{(1)}, L^{(0)}; \Lambda_{0,\mathrm{nov}}) = F^0 C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}})$. (See Subsection 3.7.4 [FOOO1].) Similarly, we have the coboundary operator $\delta_{b'_1,b'_0}$ on $C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{0,\mathrm{nov}}) = F^0 C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\mathrm{nov}})$, where we put $b'_i := \psi^{(i)}_* b_i$.

Let $\delta > 0$ be given. We consider any Hamiltonian isotopy $\phi_{H^{(0)}}$, $\phi_{H^{(1)}}$ generated by $H^{(0)}$, $H^{(1)}$ respectively such that $\phi^1_{H^{(i)}} = \psi^{(i)}$,

$$L^{(0)\prime} = \psi^{(0)}(L^{(0)}), \quad L^{(1)\prime} = \psi^{(1)}(L^{(1)}) \tag{6.9}$$

and

$$\mathrm{leng}(\phi_{H^{(0)}}) + \mathrm{leng}(\psi_{H^{(1)}}) \le \mathrm{dist}(L^{(0)}, L^{(0)\prime}) + \mathrm{dist}(L^{(1)}, L^{(1)\prime}) + \delta. \tag{6.10}$$

Denote

$$\nu_- = E^-(H^{(0)}) + E^+(H^{(1)}), \quad \nu_+ = E^+(H^{(0)}) + E^-(H^{(1)})$$

and

$$\nu := \nu_- + \nu_+ = \|H^{(0)}\| + \|H^{(1)}\| = \mathrm{leng}(\phi_{H^{(0)}}) + \mathrm{leng}(\phi_{H^{(1)}}).$$

We note that we can make ν as close to ν_0 in (6.5) as we want. See Remark 6.5.

We construct a filtered A_∞ bimodule homomorphism

$$\phi : C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}) \to C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\mathrm{nov}}).$$

One such construction is provided in in [FOOO1]. See (6.5.14) and (6.5.15) therein.

However, we would like to have an additional property that is required in Theorem 6.2 above. In [FOOO1], we used the moduli spaces of solutions for (4.6), which is the equation of moving Lagrangian boundary value problem. In this article we

use the moduli spaces of solutions u for (5.1), instead of (4.6), with $\rho = \rho_+$ such that

$$\lim_{\tau\to-\infty} u(\tau,\cdot) = \ell_p,\ \lim_{\tau\to+\infty} u(\tau,\cdot) = \mathfrak{g}^+_{H^{(0)},H^{(1)}}(\ell_{q''}),\ \Phi_{H^{(0)},H^{(1)}}(w\# u) \sim w''$$

as in Proposition 5.5. Then by identifying Crit $\mathcal{A}_{\widehat{H},\ell_a}$ with Crit $\mathcal{A}_{\ell''_a}$ we obtain a filtered A_∞ bimodule homomorphism ϕ in a way similar to Lemma 5.3.25 and Lemma 5.3.8 in [FOOO1]. The filtered A_∞ bimodule homomorphism induces a morphism of cochain complexes, which we also denote by ϕ by an abuse of notation:

$$\phi : C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}) \to C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\mathrm{nov}}). \tag{6.11}$$

Similarly, we use the moduli spaces of solutions u for (5.1) with $\rho = \rho_-$ such that

$$\lim_{\tau\to-\infty} u(\tau,\cdot) = \mathfrak{g}^+_{H^{(0)},H^{(1)}}(\ell_{p''}),\ \lim_{\tau\to+\infty} u(\tau,\cdot) = \ell_q,\ \Phi^{-1}_{H^{(0)},H^{(1)}}(w'')\# u \sim w,$$

to obtain

$$\phi' : C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\mathrm{nov}}) \to C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}). \tag{6.12}$$

Since we choose ℓ_a in such a way that $c(\widehat{H}, \ell_a) = 0$ for all $a \in \pi_0(\Omega(L^{(0)}, L^{(1)}))$, Proposition 5.5 implies that these composition satisfies

$$\phi : F^\lambda C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}) \to F^{\lambda-\nu_-} C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\mathrm{nov}}).$$

Similarly, we obtain

$$\phi' : F^\lambda C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\mathrm{nov}}) \to F^{\lambda-\nu_+} C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}).$$

This leads to the chain map

$$\phi' \circ \phi : F^\lambda C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}) \longrightarrow F^{\lambda-\nu} C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}). \tag{6.13}$$

Equivalently, we can rewrite these into the chain maps

$$T^{\nu_-}\phi : F^\lambda C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}) \to F^\lambda C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\mathrm{nov}})$$

and

$$T^{\nu_+}\phi' : F^\lambda C(L^{(1)\prime}, L^{(0)\prime}; \Lambda_{\mathrm{nov}}) \to F^\lambda C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}).$$

Setting $\lambda = 0$, we have

$$(T^{\nu_-}\phi)_* : HF((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{0,\mathrm{nov}}) \to HF((L^{(1)\prime}, b'_1), (L^{(0)\prime}, b'_0); \Lambda_{0,\mathrm{nov}})$$

and

$$(T^{\nu_+}\phi')_* : HF((L^{(1)\prime}, b'_1), (L^{(0)\prime}, b'_0); \Lambda_{0,\mathrm{nov}}) \to HF((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{0,\mathrm{nov}})$$

respectively.

We denote

$$\mathfrak{i} : F^\lambda C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}) \hookrightarrow F^{\lambda-\nu} C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}})$$

the inclusion induced homomorphism.

Lemma 6.4. *The two maps*

$$(T^{\nu_+}\phi') \circ (T^{\nu_-}\phi),\ T^\nu \mathfrak{i} : F^\lambda C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}}) \longrightarrow F^\lambda C(L^{(1)}, L^{(0)}; \Lambda_{\mathrm{nov}})$$

are chain homotopic to each other.

Proof. This last statement follows from the arguments in p.390-391 [FOOO1], and also from the explicit energy bound (5.16) in Proposition 5.6 for solutions u of (5.1) with $\widehat{H}$ and $\rho = \rho_K$, $0 \le K < \infty$, which are used to define the chain homotopy map. Recall that ρ_K extends smoothly to $K = 0$ as the constant function zero. Then the moduli spaces of solutions of (5.1) with $\rho = \rho_K$ in (5.7) defines a chain homotopy between $\phi' \circ \phi$ and the identity.

As for the filtrations, we apply (5.15) in Proposition 5.6 to all the elements in the associated parameterized moduli space defining the chain homotopy and find that the energy loss is bounded by ν for all K. This proves that $\phi' \circ \phi$ is chain homotopic to $\mathfrak{i}$ as a map

$$F^\lambda C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov}) \to F^{\lambda-\nu} C(L^{(1)}, L^{(0)}; \Lambda_{\rm nov}).$$

□

From now on, we consider the case that $\lambda = 0$. Then we have

$$(T^{\nu_+}\phi')_* \circ (T^{\nu_-}\phi)_* = T^\nu \tag{6.14}$$

where

$$T^\nu : HF((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{0,\rm nov}) \to HF((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{0,\rm nov})$$

is the map $x \mapsto T^\nu x$.

Since

$$(T^{\nu_-}\phi)_* : HF((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{0,\rm nov}) \to HF((L^{(1)\prime}, b_1'), (L^{(0)\prime}, b_0'); \Lambda_{0,\rm nov})$$

and

$$(T^{\nu_+}\phi')_* : HF((L^{(1)\prime}, b_1'), (L^{(0)\prime}, b_0'); \Lambda_{0,\rm nov}) \to HF((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{0,\rm nov})$$

are $\Lambda_{0,\rm nov}$-module homomorphisms, we have, for any $\lambda > 0$,

$$(T^{\nu_-}\phi)_* : T^\lambda HF((L^{(1)} b_1), (L^{(0)}, b_0); \Lambda_{0,\rm nov}) \to T^\lambda HF((L^{(1)\prime}, b_1'), (L^{(0)\prime}, b_0'); \Lambda_{0,\rm nov})$$

and

$$(T^{\nu_+}\phi')_* : T^\lambda HF((L^{(1)\prime}, b_1'), (L^{(0)\prime}, b_0'); \Lambda_{0,\rm nov}) \to T^\lambda HF((L^{(1)} b_1), (L^{(0)}, b_0); \Lambda_{0,\rm nov}).$$

Since $(T^{\nu_+}\phi')_* \circ (T^{\nu_-}\phi)_*$ is equal to the multiplication by T^ν, the minimal number of generators of Im $(T^{\nu_+}\phi')_* \circ (T^{\nu_-}\phi)_*$ is equal to $a + b(\lambda+\nu)$ if $\lambda + \nu \notin \{\lambda_{\downarrow,i} \mid i = 1, \dots, b\}$. On the other hand, the minimal number of generators of $T^\lambda HF((L^{(1)\prime}, b_1'), (L^{(0)\prime}, b_0'); \Lambda_{0,\rm nov})$ is equal to $a + b'(\lambda)$ if $\lambda \notin \{\lambda'_{\downarrow,i} \mid i = 1, \dots, b'\}$. Here

$$b'(\lambda) = \#\{i \mid \lambda'_{\downarrow,i} \ge \lambda\}.$$

Therefore we have

$$a + b(\lambda+\nu) \ge a + b'(\lambda)$$

for $\lambda \notin \{\lambda_{\downarrow,i} - \nu \mid i = 1, \dots, b\} \cup \{\lambda'_{\downarrow,i} \mid i = 1, \dots b'\}$ cf. Lemma 6.5.31 in [FOOO1]. This implies that $\lambda'_{\downarrow,i} \ge \lambda_{\downarrow,i} - \nu$ whenever $\lambda_{\downarrow,i} > \nu$.

Since this holds for all Hamiltonian isotopies $\phi_{H^{(0)}}$, $\phi_{H^{(1)}}$ satisfying (6.9), (6.10) and for any $\delta > 0$, we obtain

$$\text{if } \lambda_{\downarrow,i} > \nu, \quad \lambda_{\downarrow,i} \le \nu + \lambda'_{\downarrow,i}. \tag{6.15}$$

By changing the role of $L^{(1)}, L^{(0)}$ with $L^{(1)\prime}, L^{(0)\prime}$ we also obtain

$$\text{if } \lambda'_{\downarrow,i} > \nu, \quad \lambda'_{\downarrow,i} \le \nu + \lambda_{\downarrow,i}. \tag{6.16}$$

Theorem 6.2 follows. □

Remark 6.5. With given fixed $L^{(0)\prime} \in \mathfrak{Iso}(L^{(0)})$ and $L^{(1)\prime} \in \mathfrak{Iso}(L^{(1)})$, we may consider *all* possible Hamiltonian isotopy with given end points and take the infimum of $\mathrm{leng}(\phi_{H^{(0)}}) + \mathrm{leng}(\phi_{H^{(1)}})$ over all $H^{(0)}$ and $H^{(1)}$ such that

$$\phi^1_{H^{(0)}}(L^{(0)}) = L^{(0)\prime}, \quad \phi^1_{H^{(1)}}(L^{(1)}) = L^{(1)\prime}.$$

In this way, we can make $\mathrm{leng}(\phi_{H^{(0)}}) + \mathrm{leng}(\phi_{H^{(1)}})$ as close to the sum

$$\mathrm{dist}(L^{(0)}, L^{(0)\prime}) + \mathrm{dist}(L^{(1)}, L^{(1)\prime})$$

as we want.

7. Torsion threshold and displacement energy

As we mentioned, the torsion exponents of the Floer cohomology have some information on the displacement energy of Lagrangian submanifolds. We introduce the following notion to describe a relation between the torsion exponents and the displacement energy.

Definition 7.1. Let $L^{(1)}, L^{(0)}$ be weakly unobstructed Lagrangian submanifolds after bulk deformations. Let

$$(\mathbf{b}_1, \mathbf{b}_0) \in \mathcal{M}_{\mathrm{weak,def}}(L^{(1)}) \times_{(\pi_{\mathrm{amb}}, \mathfrak{PO})} \mathcal{M}_{\mathrm{weak,def}}(L^{(0)})$$

as in (6.1). Suppose $HF((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{\mathrm{nov}}) = 0$. We denote by λ_i its torsion exponents defined by (6.2).

(1) We define

$$\mathfrak{T}((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0)) = \max_i \lambda_i$$

and call it the *torsion threshold* of the pair $(L^{(0)}, L^{(1)})$ relative to $(\mathbf{b}_1, \mathbf{b}_0)$.

(2) We define

$$\mathfrak{T}(L^{(1)}, L^{(0)}) = \sup_{(\mathbf{b}_1, \mathbf{b}_0)} T((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0))$$

and call it the *torsion threshold* of the pair $(L^{(0)}, L^{(1)})$.

When $HF((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{\mathrm{nov}}) \neq 0$ for some $(\mathbf{b}_1, \mathbf{b}_0)$, we define

$$\mathfrak{T}(L^{(1)}, L^{(0)}) = \infty.$$

(3) In the case $b_i \in \mathcal{M}_{\mathrm{weak}}(L^{(i)})$, we define $\mathfrak{T}((L^{(1)}, b_1), (L^{(0)}, b_0))$ and $\mathfrak{T}(L^{(1)}, L^{(0)})$ in a similar manner. Here the supremum is taken over the set

$$(b_1, b_0) \in \{(b_1, b_0) \in \mathcal{M}_{\mathrm{weak}}(L^{(1)}) \times \mathcal{M}_{\mathrm{weak}}(L^{(0)}) \mid \mathfrak{PO}(b_1) = \mathfrak{PO}(b_0)\}.$$

(4) We just denote $\mathfrak{T}((L, \mathbf{b}), (L, \mathbf{b}))$ and $\mathfrak{T}(L, L)$ by $\mathfrak{T}(L, \mathbf{b})$ and $\mathfrak{T}(L)$ respectively.

We now specialize the energy estimate in the previous section to the particular case

$$(L^{(0)}, L^{(1)}) = (L, L), \quad (L^{(0)\prime}, L^{(1)\prime}) = (L, \psi^{(1)}(L))$$

with the displacing condition

$$L \cap \psi^{(1)}(L) = \emptyset. \tag{7.1}$$

Then the following theorem relating the displacement energy and the torsion threshold of L is a special case of Theorem J. For readers' convenience, we give its proof which specializes the proof of Theorem J to this particular context.

Theorem 7.2. *Let L be a relatively spin closed Lagrangian submanifold of (X,ω). Suppose that L is weakly unobstructed after bulk deformation and displaceable. We denote by $e(L)(=e^X(L))$ its displacement energy. Let the torsion threshold of $HF((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{0,nov})$ be positive, i.e. assume $\mathfrak{T}(L,\mathbf{b})>0$. Then we have*

$$e(L)\geq \mathfrak{T}(L,\mathbf{b})$$

for any $\mathbf{b}\in\mathcal{M}_{\rm weak,def}(L)$. In particular, we have $e(L)\geq\mathfrak{T}(L)$.

Proof. Suppose to the contrary that there exist a sufficiently small $\delta>0$ and an element $\mathfrak{b}\in\mathcal{M}_{\rm weak,def}(L)$ such that

$$e(L)<\mathfrak{T}(L,\mathbf{b})-\delta.$$

Pick a Hamiltonian H and its associated Hamiltonian isotopy ϕ_H such that

$$\phi_H^1(L)\cap L=\emptyset,\quad \|H\|\leq e(L)+\delta.$$

In particular, we also have

$$\|H\|<\mathfrak{T}(L,\mathbf{b}).$$

Now we recall from (6.13) that $\phi'_*\circ\phi_*$ restricts to

$$T^\lambda HF((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{0,\rm nov})\to T^{\lambda-\|H\|}HF((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{0,\rm nov}).$$

and satisfies

$$\phi'_*\circ\phi_*=(\phi'\circ\phi)_*=\mathfrak{i}_*$$

as a map

$$T^\lambda HF((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{0,\rm nov})\to T^{\lambda-\|H\|}HF((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{0,\rm nov}) \tag{7.2}$$

for all $\lambda\in\mathbb{R}$.

We now specialize to the case $\lambda=\|H\|$. In this case,

$$T^{\lambda-\|H\|}HF((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{0,\rm nov})=HF((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{0,\rm nov}).$$

Since $\lambda<\mathfrak{T}(L,\mathbf{b})$, the image of the inclusion-induced map

$$\mathfrak{i}_*:T^\lambda HF((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{0,\rm nov})\to HF((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{0,\rm nov})$$

is not trivial by the definition of $\mathfrak{T}(L,\mathbf{b})$.

On the other hand, $HF((\phi_H^1(L),\phi_{H*}^1\mathbf{b}),(L,\mathbf{b});\Lambda_{0,\rm nov})=\{0\}$ by the hypothesis $L\cap\phi_H^1(L)=\emptyset$ and hence $\phi_*=0=\phi'_*$ which implies $\phi'_*\circ\phi_*=0$.

Therefore the equality $\phi'_*\circ\phi_*=(\phi'\circ\phi)_*=\mathfrak{i}_*$ with $\lambda=\|H\|$ in (7.2) gives rise to a contradiction. This finishes the proof. □

8. Displacement of polydisks inside cylinders in high dimensions

In this section, we consider the situation of [H] in any dimension. Namely, we prove Theorem 1.3 and Theorem 1.4 stated in Section 1.

We recall the polydisks in $\mathbb{C}^n$ denoted by

$$D(a_1,a_2,\dots,a_n)=\{(z_1,\dots,z_n)\in\mathbb{C}^n\mid \pi|z_1|^2<a_1,\dots,\pi|z_n|^2<a_n\}$$

where $a_1\leq a_2\leq\cdots\leq a_n$. Hind considers only the case when $n=2$. We also denote the cylinder over the disk $|z_1|^2\leq(a_1+\varepsilon)/\pi$ by

$$Z_{1,n-1}(a_1+\varepsilon)=\{(z_1,\dots,z_n)\mid\pi|z_1|^2<a_1+\varepsilon\}$$

for $0<\varepsilon<1$.

Theorem 8.1 (Theorem 1.3)**.** *Suppose that $S > 1$ and $0 < \varepsilon < 1$. Let $Z_{1,n-1} = Z_{1,n-1}(1+\varepsilon)$. Then we have*

$$S \leq e^{Z_{1,n-1}}(D(1, S, \ldots, S)).$$

Proof. We prove this by contradiction. Suppose $e^{Z_{1,n-1}}(D(1, S, \cdots, S)) < S$ and so

$$e^{Z_{1,n-1}}(D(1, S, \ldots, S)) < S - \delta$$

for some small $\delta > 0$. By definition of $e^{Z_{1,n-1}}(D(1, S, \ldots, S))$, there exists a Hamiltonian H on $Z_{1,n-1}$ such that

$$\phi_H^1(D(1, S, \ldots, S)) \cap D(1, S, \ldots, S) = \emptyset, \quad \operatorname{supp} \phi_H \subset Z_{1,n-1}$$

and

$$\|H\| \leq e^{Z_{1,n-1}}(D(1, S, \ldots, S)) + \delta < S, \tag{8.1}$$

where the inequality comes from the standing hypothesis. Since

$$\operatorname{supp} \phi_H \subset \operatorname{Int} Z_{1,n-1}(1+\varepsilon)$$

is compact, we can symplectically embed

$$D(1, S, \ldots, S) \subset S^2(1+\varepsilon') \times \underbrace{S^2(\lambda) \times \cdots \times S^2(\lambda)}_{(n-1) \text{ times}} =: X$$

together with the image of $D(1, S, \ldots, S)$ by the isotopy $\phi_H^t, 0 \leq t \leq 1$, for some ε' with $0 < \varepsilon < \varepsilon'$ and sufficiently large $\lambda > 0$. For the later purpose, we take ε' and λ which satisfy $0 < \varepsilon < \varepsilon' < 1$ and $\lambda > 2S$.

We consider a circle $S^1(S) \subset S^2(\lambda)$ which divides $S^2(\lambda)$ into two domains of areas S and $\lambda - S$ respectively. Then we consider the torus

$$L = L\left(\frac{1+\varepsilon'}{2}, S, \ldots, S\right) = S^1\left(\frac{1+\varepsilon'}{2}\right) \times S^1(S) \times \cdots \times S^1(S),$$

which is a subset of $D(1, S, \ldots, S)$ because $\varepsilon' < 1$. This torus L is displaceable by ϕ_H inside $X = S^2(1+\varepsilon') \times \underbrace{S^2(\lambda) \times \cdots \times S^2(\lambda)}_{(n-1) \text{ times}}$ since $D(1, S, \ldots, S)$ is so. Therefore we have $e^X(L) \leq \|H\|$ which follows from the definition of e^X. In particular, by (8.1) we have

$$e^X(L) < S. \tag{8.2}$$

On the other hand, we know that the torus

$$L = L\left(\frac{1+\varepsilon'}{2}, S, \ldots, S\right) \subset S^2(1+\varepsilon') \times \underbrace{S^2(\lambda) \times \cdots \times S^2(\lambda)}_{(n-1) \text{ times}}$$

is one of the toric fiber. By Proposition 4.3 of [FOOO2] it is weakly unobstructed (i.e., $\mathcal{M}_{\text{weak}}(L) \neq \emptyset$) and we can choose a weak bounding cochain $b \in \mathcal{M}_{\text{weak}}(L)$ as $b = 0$.

Now it remains to show

Lemma 8.2. *Choose the weak bounding cochain $b = 0$. Then we have*

$$\mathfrak{T}(L, 0) \geq S.$$

Proof. By a result of [CO] the Maslov index 2 disks are completely classified. They consist of the obvious ones coming from the the upper and lower hemispheres of $S^2(1+\varepsilon')$ which have equal areas $\frac{1+\varepsilon'}{2}$, and those two domains coming from $S^2(\lambda)\setminus S^1(S)$. The coboundary map $\mathfrak{m}_1$ of the Floer cochain complex of L are contributed by these disks. Since $\varepsilon' < 1 < S$, the holomorphic disks with the minimal area are the first two disks

$$D^2_{\pm}\left(\frac{1+\varepsilon'}{2}\right) \times \{pt\} \times \cdots \times \{pt\} \subset X,$$

which cancel each other in the operation of $\mathfrak{m}_1$. See Case I-a in Subsection 3.7.6 [FOOO1] and Theorem 1.3 [FOOO4] for this cancellation argument. The area of the next smallest area disk is S because $\lambda > 2S$. We have $(n-1)$ holomorphic disks with area S:

$$\{pt\} \times \cdots \times D^2_l(S) \times \{pt\} \times \cdots \times \{pt\} \subset X, \quad l = 2, \ldots, n,$$

where $D^2_l(S)$ is the disk with area S bounding the circle $S^1(S)$ in the l-th factor $S^2(\lambda)$ of X. We note that such holomorphic disks contribute to $\mathfrak{m}_1$ with the same sign. (See Theorem 11.1 (3) in [FOOO2] for more general result on orientations of moduli spaces of the Maslov index 2 disks in toric manifolds.) In particular, these holomorphic disks do not cancel each other. Then the argument similar to one of Case I-b in Subsection 3.7.6 [FOOO1] shows that they produce a torsion part $\Lambda_{0,\mathrm{nov}}/T^S\Lambda_{0,\mathrm{nov}}$ in the Floer cohomology of L. It follows that

$$\mathfrak{T}(L,0) \geq S.$$

□

Combining (8.2) and this lemma, we obtain

$$e^X(L) < \mathfrak{T}(L,0).$$

But this contradicts to Theorem 7.2 and finishes the proof of Theorem 8.1. □

By a similar argument, we can show the following variant of Theorem 8.1. We consider the domain

$$D_{n-k,k}(1,S) := D^2(1)^{n-k} \times B^{2k}(kS)$$

for $k = 1, \ldots, n-1$. Here $B^{2k}(kS)$ is the ball in $\mathbb{C}^k$ of redius r with the Gromov width $\pi r^2 = kS$.

Theorem 8.3 (Theorem 1.4)**.** *Suppose that $S > 1$ and $0 < \varepsilon < 1$. Let $Z = Z_{n-k,k}(1+\varepsilon) = D^2(1+\varepsilon)^{n-k} \times \mathbb{C}^k$. Then we have*

$$S \leq e^{Z_{n-k,k}}(D_{n-k,k}(1,S)).$$

Proof. The proof will be the same as that of Theorem 8.1 with the following modifications. We again prove this by contradiction. Suppose $e^{Z_{n-k,k}}(D_{n-k,k}(1,S)) < S$ and choose $\delta > 0$ and H as before so that

$$e^{Z_{n-k,k}}(D_{n-k,k}(1,S)) < S - \delta$$

and

$$\phi^1_H(D_{n-k,k}(1,S)) \cap D_{n-k,k}(1,S) = \emptyset, \quad \operatorname{supp}\phi_H \subset Z_{n-k,k}(1+\varepsilon),$$

and

$$\|H\| \leq e^{Z_{n-k,k}}(D_{n-k,k}(1,S)) + \delta < S.$$

Then we can symplectically embed

$$D_{n-k,k}(1,S) \subset S^2(1+\varepsilon')^{n-k} \times \mathbb{C}P^k(\lambda) =: X$$

together with the image of $D_{n-k,k}(1,S)$ by the isotopy $\phi_H^t, 0 \le t \le 1$, for some ε' with $0 < \varepsilon < \varepsilon' < 1$ and sufficiently large $\lambda > 0$. Then we consider the torus

$$\begin{aligned} L &= S^1\left(\frac{1+\varepsilon'}{2}\right)^{n-k} \times S^1(S)^k \\ &\subset S^2(1+\varepsilon')^{n-k} \times B^{2k}(kS) \subset S^2(1+\varepsilon')^{n-k} \times \mathbb{C}P^k(\lambda). \end{aligned}$$

The torus L is also contained in $D_{n-k,k}(1,S)$ because $\varepsilon' < 1$. Note that L is one of the toric fiber in $X = S^2(1+\varepsilon')^{n-k} \times \mathbb{C}P^k(\lambda)$. The rest of the proof is similar to one of Theorem 8.1. So we omit it. □

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